Coloured Cyclic Operads

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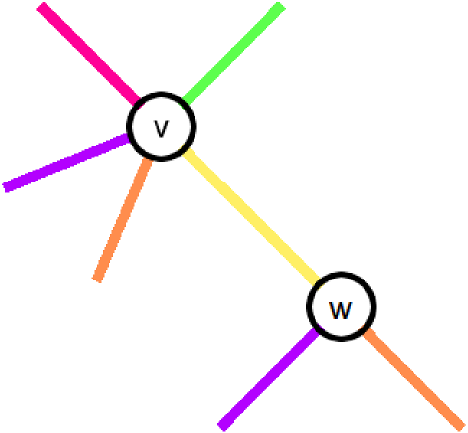
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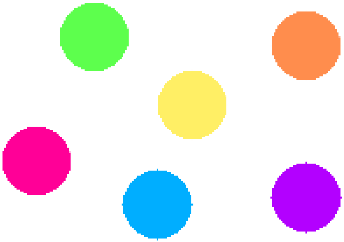
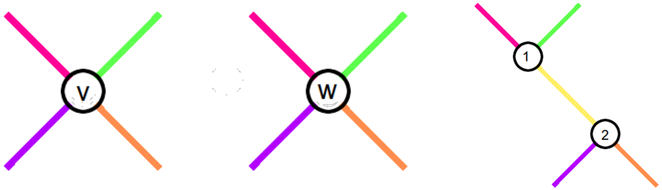
Definition

Let $ob(\mathcal{C})$ be a class of objects, or colours.



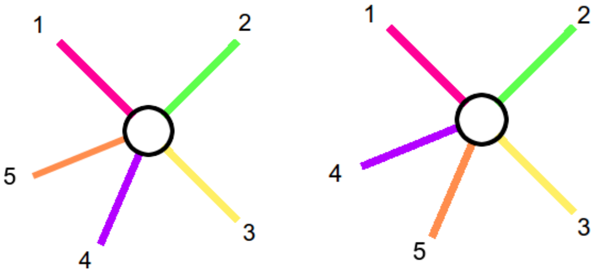
Definition

For each profile over $ob(\mathcal{C})$, $\underline{c} = (c_1, \ldots, c_n)$, there is a class $\mathcal{C}(c_1, \ldots c_n)$.



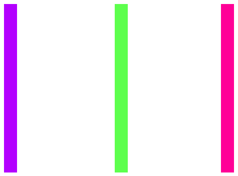
Definition

For any profile $\underline{c}$ and permutation $\sigma \in \Sigma_n$, there is a right action of the symmetric group, a bijection $\mathcal{C}(c_1, \ldots c_n) \to \mathcal{C}(c_{\sigma(1)}, \ldots c_{\sigma(n)})$



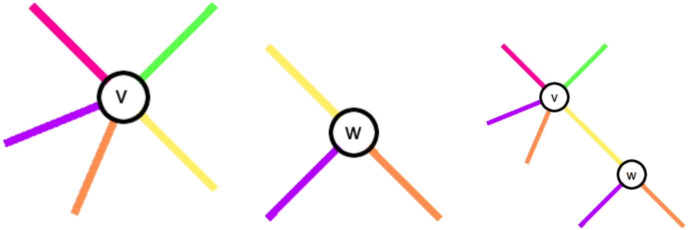
Definition

For each colour $c \in ob(\mathcal{C})$ there is an identity element $\eta_c \in \mathcal{C}(c, c)$



Definition

There is an associative, unital, and equivariant composition operation which identifies two edges of the same colour.



Dagger categories

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This is a cyclic operad where $\mathcal{C}(\underline{c})$ is empty unless $\underline{c}$ has length 2

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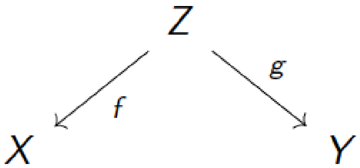

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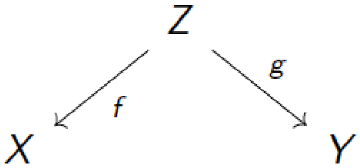
$$X \xleftarrow{\ f\ } Z \xrightarrow{\ g\ } Y$$

In a similar way, we can create multispans...

Multispans

Let $(X_i)_{i=0}^{n}$ be objects in a category $\mathcal{C}$, along with morphisms $f_i : X_0 \to X_i$. Then $(X_1, \ldots X_m; X_0)$ is a multispan.



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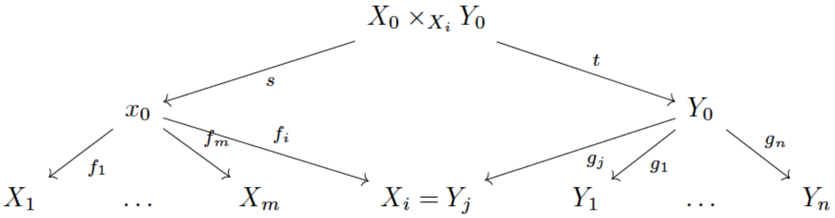

Definition (Composition of multispans)

Consider spans $(X_1, \ldots X_m; X_0)$ and $Y_1, \ldots Y_n; Y_0)$ such that some $X_i = Y_j$. Then we can find a span $(X_1, \ldots \hat{X}_i, \ldots X_m, Y_1, \ldots \hat{Y}_i, \ldots Y_n; X_0 \times_{X_i} Y_0)$, where $(X_0 \times_{X_i} Y_0), s, t)$ is the pullback of X_0 and Y_0 over $X_i = Y_j$. The internal morphisms will be given by compositions $f_i \circ s$ and $g_i \circ t$.

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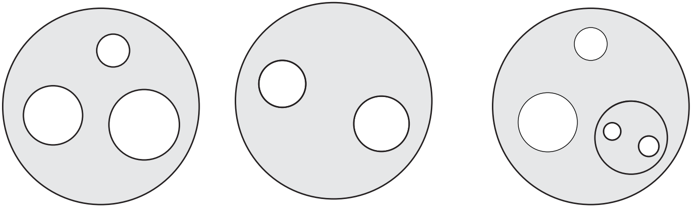
Framed little discs



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Let D^n be the open unit disc. Then for all $k \in \mathbb{N}$, let $\mathcal{D}_n(k)$ be the space of embeddings of k disjoint discs into a disc,

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where f is a composition of translations, dilations and rotations. Let f, g_i be such maps. Then composition of maps is defined by compositions of disjoint unions of maps:

$$\circ(f, g_1, \dots g_k) = \coprod_{n_1+\dots n_k} D^n \xrightarrow{g_1 \sqcup \dots \sqcup g_k} \coprod_k D^n \xrightarrow{f} D^n$$

Surface operad