A Mapping Class Group Operad

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Motivation

- Explore different conceptions of a surface operad
- Design an operad starting from dendroidal sets rather than going the other way
- Define a surface operad useful in Grothendiek-Teichmuller theory

1. The Mapping Class Groupoid
2. Dendroidal sets
3. Infinity operads

Elementary surfaces

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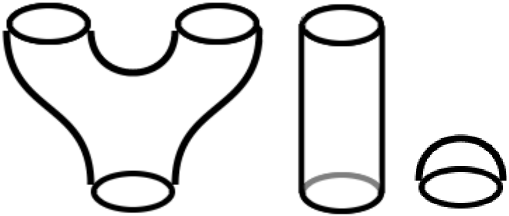
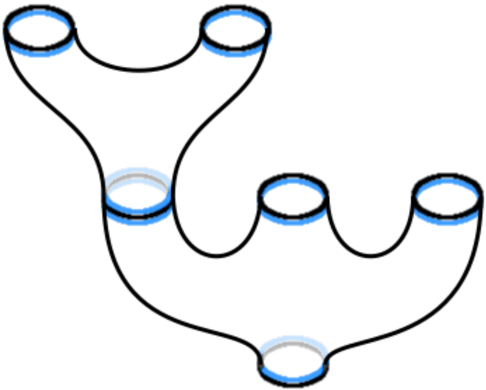


Gluing surfaces



$[0, \varepsilon) \times S^1 \hookrightarrow$ Surface

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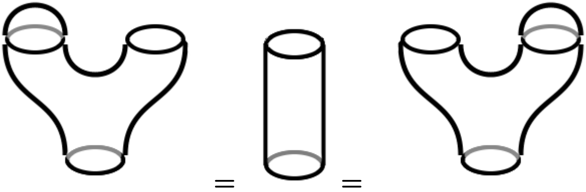

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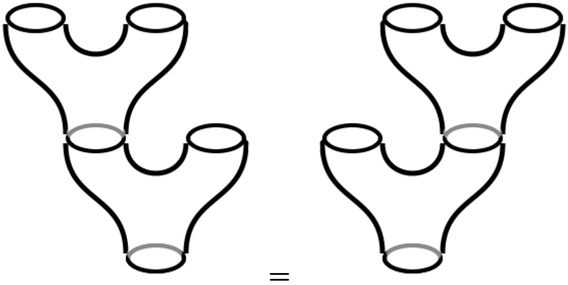

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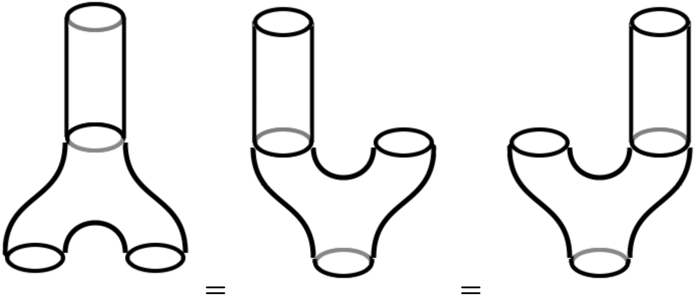
3. If $\mathcal{E}_{n+1}$ is any such surface, then $\mathcal{E}_{n+1} \circ_i C = \mathcal{E}_{n+1} = C \circ \mathcal{E}_{n+1}$



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Definition (Isotopic)

Let A and B be surfaces, with $f, g : A \to B$ diffeomorphisms between them. If there is a homotopy $H : f \to g$ which is, at each stage, a homeomorphism, then f and g are said to be isotopic.

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Example (Cylinder)

The cylinder has genus 0 and two boundary components. Morphisms correspond to the number of Dehn twists that have been done. Hence, $S_0^{1+1} = \mathbb{Z}$.

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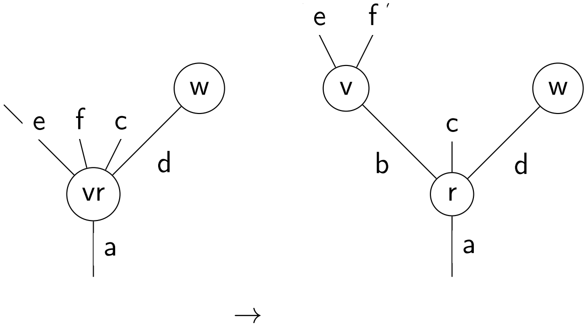
The category Ω contains rooted trees as objects, with maps between them as morphisms. The maps are generated by:

- inner coface maps



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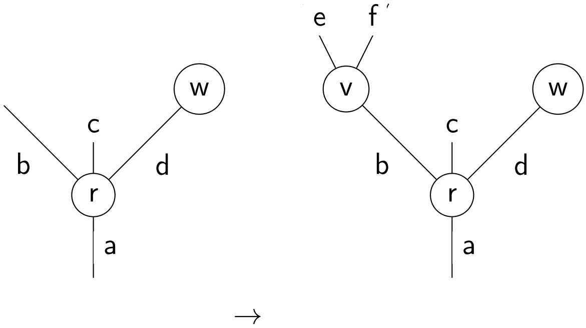

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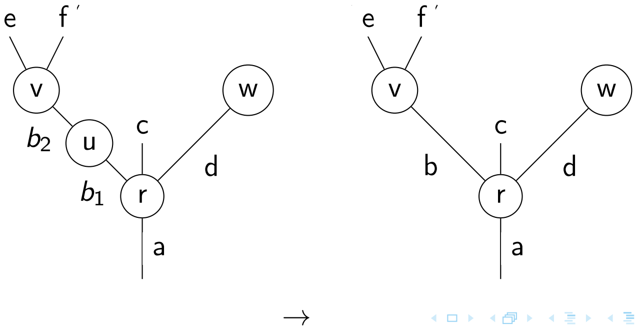

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- An outer coface map $\partial_v : T/v \to T$ maps to the functor generated by the relevant inclusion map
- A degeneracy map $\sigma_e : T\backslash e \to T$ maps to the functor which inserts an inverse Dehn twist corresponding to the extra edge

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Let X be a dendroidal set. Note that $\Omega[T]$ is the representable functor at a tree T and $\Lambda_e[T]$ is the inner horn, a dendroidal set formed by removing an edge e from a tree T.

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Definition (Operad)

Let X be a dendroidal set. If the filler g exists and is unique, then X is a strict quasi-operad, otherwise known as just an operad.

Mapping class infinity operad?

Q. Is our mapping class group dendroidal set an infinity operad?

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Theorem

Let G be a dendroidal set. If there exists a dendroidal groupoid structure on G, then the underlying dendroidal set satisfies the inner Kan condition.

Further research

There are a number of directions to go from here.

- Since surfaces do not necessarily have in and out boundaries, we should consider the cyclic case. That is, quasi cyclic operads. I am currently working on the cyclic analogue to dendroidal sets.
- Surfaces may have genus greater than zero. Therefore higher genus analogues of operads should be considered, such as modular operads.
- The Teichmuller tower uses the profinite completions of the mapping class groups, so it would be interesting to consider the corresponding operad.

Acknowledgements

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