# The classification of 8-dimensional E-manifolds and Sullivan's conjecture

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## E-manifolds

**Definition.** $M$ is an *E-manifold* if $H_{2k+1}(M;\mathbb{Z}) \cong 0$ for all $k$.

We will consider smooth, closed, simply-connected, oriented E-manifolds.

Our goal is to give a method to classify 8-dimensional E-manifolds up to diffeomorphism.

**Examples.**

- *Homotopy spheres:* $\Theta_8 \cong \mathbb{Z}_2$
- *The projective spaces* $\mathbb{C}P^4$ *and* $\mathbb{H}P^2$
- *Complete intersections.*

## Complete intersections

**Definition.** A subspace $M \subset \mathbb{C}P^{k+4}$ is a *complete intersection* of (complex) dimension 4, if

- there are homogeneous polynomials $f_i \in \mathbb{C}[x_1, x_2, \dots, x_{k+5}]$ for $i = 1, 2, \dots, k$ such that
$$M = \left\{ \underline{x} \in \mathbb{C}P^{k+4} \;\middle|\; f_i(\underline{x}) = 0 \text{ for all } i \right\}$$
- $M$ is a smooth 8-dimensional submanifold of $\mathbb{C}P^{k+4}$

**Theorem** (Thom)**.** *The diffeomorphism class of $M$ depends only on the degrees of the polynomials $f_i$.*

**Conjecture** (Sullivan)**.** *Two complete intersections are diffeomorphic if and only if they have the same total degree (= the product of the degrees), Pontryagin-classes and Euler-characteristic.*

**Theorem** (Fang-Klaus)**.** *The Sullivan-conjecture holds up to homeomorphism for 4-dimensional complete intersections.*

**Theorem** (Crowley-N.)**.** *The Sullivan-conjecture holds for 4-dimensional complete intersections for which either $w_2 = 0$ or $w_2 \neq 0$ and $w_4 \neq 0$.*

## The classification of E-manifolds

The classification of E-manifolds will consist of 4 steps:

1. Define an action of $\theta(r, w)$ on $E(r, w)$
2. Compute the group $\theta(r, w)$
3. Classify the orbits of the action
4. Compute the stabilizers of the orbits

## 1. Definitions

**Definition.** For an $r \geq 0$ and $w : \mathbb{Z}^r \to \mathbb{Z}_2$ let

$$E(r, w) = \left\{ (M, \varphi) \,\middle|\, \begin{array}{c} M \text{ is an E-manifold} \\ \varphi : H_2(M) \cong \mathbb{Z}^r \\ w_2(M) = w \end{array} \right\} \Big/ \sim$$

where the equivalence relation $\sim$ is orientation-preserving diffeomorphism, compatible with $\varphi$.

**Remark.** *If $w, w' : \mathbb{Z}^r \to \mathbb{Z}_2$ are both non-zero, then there is an isomorphism $I : \mathbb{Z}^r \to \mathbb{Z}^r$ such that $w' = w \circ I$, therefore $E(r, w) \cong E(r, w')$.*

**Definition.** For an $r \geq 0$ and $w : \mathbb{Z}^r \to \mathbb{Z}_2$ let

$$T = T(w) \approx D_1 \natural D_2 \natural \ldots \natural D_r$$

where each $D_i$ is a $D^6$-bundle over $S^2$, such that $D_i$ is trivial iff $w(e_i) = 0$, where $e_1, e_2, \dots, e_r$ is the standard basis of $\mathbb{Z}^r$ (there is a unique nontrivial bundle, because $D^6$-bundles over $S^2$ correspond to elements of $\pi_2(BSO_6) \cong \mathbb{Z}_2$).

**Definition.** There is a monoid structure on $E(r, w)$, the operation is "connected sum along the 2-skeleton" (denoted by $\#_2$), defined as follows:

$\varphi$ determines an embedding $T \hookrightarrow M$:

- The basis elements $e_1, e_2, \dots, e_r \in \mathbb{Z}^r \cong H_2(M) \cong \pi_2(M)$ are represented by disjoint embedded spheres $S^2 \hookrightarrow M$
- The normal bundle of an $S^2$ is an element of $\pi_2(BSO_6) \cong \mathbb{Z}_2$, classified by the Stiefel-Whitney class $w_2$
- Since $w_2(M) = w$, the tubular neighbourhood of the $i^{th}$ sphere is diffeomorphic to $D_i$
- Therefore the boundary connected sum of the tubular neighbourhoods of the spheres is an embedded copy of $T$ in $M$ (well-defined up to isotopy)

With these embeddings we define $M_1 \#_2 M_2 = (M_1 \setminus T) \cup_{\partial T} (M_2 \setminus T)$.

**Definition.**

$$\theta(r, w) = \{(\Sigma, \varphi) \in E(r, w) \mid H_4(\Sigma) = 0\} / \sim$$

**Example.** $\theta(0, 0) \cong \Theta_8 \cong \mathbb{Z}_2$.

**Proposition.** *$\theta(r, w)$ is the group of invertible elements in the monoid $E(r, w)$.*

## 2. The group $\theta(r, w)$

**Theorem** (Crowley-N.)**.**

- $\theta(r, 0) \cong \mathbb{Z}^a \oplus \mathbb{Z}_2^b$*, where* $a = 6\binom{r+2}{5}$, $b = 2\binom{r+3}{4} - \binom{r-1}{2} + 2$.
- $\operatorname{rk} \theta(r, w) = a$.
- $\theta(1, 1) \cong \mathbb{Z}_4$.

## 3. Classification of the orbits

**Theorem** (Crowley-N.)**.** *The orbits of the action of $\theta(r, w)$ on $E(r, w)$ are classified by the cohomology ring and the Pontryagin-class $p_1$. More precisely, the following are equivalent:*

- *$M_1 \approx M_2 \#_2 \Sigma$ for some $\Sigma \in \theta(r, w)$.*
- *There is a ring isomorphism $H^*(M_1) \to H^*(M_2)$ that is compatible with the maps $\varphi_i^* : H^2(M_i) \cong \mathbb{Z}^r$ and preserves $p_1$.*

Special cases were proved by Wall (for $r = 0$) and Schmitt (for any $r$ and $w = 0$).

## 4. Inertia groups

**Definition.** The *(extended) inertia group* of an E-manifold $M \in E(r, w)$ is the subgroup

$$I(M) = \{\Sigma \in \theta(r, w) \mid M \#_2 \Sigma \approx M\} \leq \theta(r, w)$$

Our main tool for calculating inertia groups is the Q-form conjecture.

### The Q-form conjecture

**Theorem** (Q-form conjecture, torsion-free case)**.** *Suppose that $q$ is even, and $M$ and $M'$ are simply-connected $2q$-manifolds. If there exists*

- *a stable bundle $\xi$ over some base space $B$*
- *a cobordism $W$ between $M$ and $M'$*
- *a map $F : W \to B$*

*such that*

- *$H_q(B)$ is torsion-free*
- *$F$, $f := F\big|_M$ and $f' := F\big|_{M'}$ are normal $(q-1)$-smoothings*
- *$Q(f) \cong Q(f')$*

*then $M \approx M'$ (in fact $W$ is cobordant to $M \times [0, 1]$, relative to the boundary).*

where

**Definition.** Let $\xi$ be a stable vector bundle over some space $B$. For a manifold $M$ the map $f : M \to B$ is a *normal $k$-smoothing*, if

- $\pi_i(f)$ is an isomorphism for $i \leq k$ and $\pi_{k+1}(f)$ is surjective (that is, $f$ is a $(k+1)$-equivalence)
- $\nu_M \cong f^*(\xi)$

**Proposition.** *Every manifold $M$ has a canonical $k$-smoothing $M \to B^k(M)$, where $B^k(M)$ is the $(k+1)^{th}$ Moore-Postnikov stage of the classifying map $M \to BO$ of the stable normal bundle $\nu_M$ of $M$.*

**Definition.** The $k^{th}$ *stage of the Moore-Postnikov factorization* of a map $f : X \to Y$ is a space $MP_k(f)$ together with maps $f_1 : X \to MP_k(f)$ and $f_2 : MP_k(f) \to Y$ such that

- $f = f_2 \circ f_1$
- $\pi_i(f_1)$ is an isomorphism for $i < k$ and $\pi_k(f_1)$ is surjective
- $\pi_i(f_2)$ is an isomorphism for $i > k$ and $\pi_k(f_2)$ is injective

**Examples.**

- *If $Y \simeq *$, then $MP_{k+1}(f) = P_k(X)$ is a Postnikov-stage of $X$.*
- *If $X \simeq *$, then $MP_k(f)$ is the $k$-connected cover of $Y$.*

**Remark.** *The existence of the normal smoothing $M \to B^k(M)$ shows that the converse of the Q-form conjecture is also true ($W$ can be chosen to be $M \times I$).*

**Definition.** Fix some even positive integer $q$ and a space $B$ with a stable bundle $\xi$ over it. If $M$ is a $2q$-dimensional manifold and $f : M \to B$ is a $(q-1)$-smoothing, then the *Q-form* of $f$ is the triple

$$Q(f) = (H_q(M), \lambda_M, f_*)$$

where

- $\lambda_M : H_q(M) \times H_q(M) \to \mathbb{Z}$ is the intersection pairing
- $f_* : H_q(M) \to H_q(B)$ is induced by $f$

### Application to inertia groups

Suppose that $M \in E(r, w)$ and $\Sigma \in \theta(r, w)$ and let $M \#_2 \Sigma$. Then $B^3(M) = B^3(M')$, let $B$ denote this space. Let $f : M \to B$ and $f' : M' \to B$ denote the canonical normal smoothings, then $Q(f) \cong Q(f')$. We can use the Q-form conjecture to decide if $\Sigma \in I(M)$:

**Proposition.** *$f$ and $f'$ are bordant over $B \iff \Sigma \in I(M)$.*

**Theorem** (Crowley-N.)**.**

- *If* $M \in E(0,0)$*, then*

$$I(M) = \begin{cases} 0 & \text{if } p_1 \text{ is divisible by } 8 \\ \theta(0,0) \cong \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

- *If* $M \in E(1,1)$*,* $\pi_3(M) \cong 0$ *and* $w_4(M) \neq 0$*, then* $I(M) = \theta(1,1) \cong \mathbb{Z}_4$
- *If* $M \in E(r,w)$*,* $\cup_2 : H^2(M) \times H^2(M) \to H^4(M)$ *is trivial and* $p_1 = 0$*, then* $I(M) = 0$
- *The subgroup* $I(M) \leq \theta(r,w)$ *only depends on* $\cup_2$ *and* $p_1$*.*

## Proof of the Sullivan-conjecture

**Theorem** (Crowley-N.)**.** *The Sullivan-conjecture holds for* 4*-dimensional complete intersections for which either* $w_2 = 0$ *or* $w_2 \neq 0$ *and* $w_4 \neq 0$*.*

*Proof.* (In the case $w_2 \neq 0$, $w_4 \neq 0$)

- Such complete intersections are in $E(1,1)$.
- If the invariants of two complete intersections agree, then they are in the same orbit of $\theta(1,1)$.
- By the previous theorem the stabilizer of this orbit is $\theta(1,1)$. Therefore the orbit has only one element, so the complete intersections are diffeomorphic.

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