Smooth parametrised Whitehead torsion and A-theory simplicial presheaf

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1 Introduction

Structure of the talk: I'm going to talk about parametrised Whitehead torsion. My talk will be in three parts. In the first part, I'll descirbe Becker-Gottlieb transfer and DWW's equivalent construction. Then in the second part I'll describe the construction of the smooth parametrised Whitehead torsion and the justification of the name. Finally, I'll go on to talk about the construction of A-theory simplicial presheaf.

Significance and relevance: [3] constructed a parametrised smooth Whitehead torsion for smooth compact manifold bundle using homotopy theoretic construction, containing information about the smooth structure. This generalises the Whitehead torsion. And this torsion is up to constant equivalent to the higher tosion of Bismut and Lott in dimensions $4k$, constructed using Morse theory and analytic method. And $\tau^{IK} = \tau^{BL}$.

Methodology: For simplicity, we restrict ourselves to smooth closed manifold M^n. In fact, all the constructions can be applied to topological compact manifolds equipped with an embedding of the n-disk bundle into the tangent bundle. To better illustrate the construction, we describe it for $M \to *$ and then explain briefly how to generalise it to parametrised version.

2 Construction in [3]

2.1 Becker-Gottlieb-Dold transfer[1]

To begin with, recall that for a fibration $p : E \to B$ with finitely dominated fibre and a map $f : E \to E$ over B, there is a pointed map $B_+ \to E_+$, s.t. the induced map on cohomology groups of $B_+ \to E_+ \to B_+$ is multiplication by the Lefschetz number Λ_f. In this sense, the transfer map serves as a variant of push forward for cohomology theory.

Now suppose $f : M \to M$. Identify a space with its suspension spectrum, then the Becker-Gottlieb transfer is given by

$$S^0 \xrightarrow{u} M_+ \wedge \hat{M}_+ \xrightarrow{1\wedge f_+ \wedge 1} M_+ \wedge M_+ \wedge \hat{M}_+ \xrightarrow{1\wedge \hat{u}} M_+ \wedge S^0.$$

$\hat{M}_+$ is the Spanier-Whitehead dual of M_+, which by Atiyah duality is homotopy equivalent to $\Sigma^{-w}\,\mathrm{Th}(NM)$ where the normal bundle is taken with respect to an embedding $M \hookrightarrow \mathbb{R}^w$. Then $\deg(p_* \circ \mathrm{tr}_f) = \Lambda_f$. In particular, if $f = id_M$, then $\Lambda_f = \chi(M)$. Becker-Gottlieb-Dold transfer is well defined up to homotopy.

For a general fibration $p : E \to B$ with finitely dominated fibre, Becker and Gottlieb constructed a fibrewise dual of $\bar{E} = E \coprod B$ and defined the transfer to be a map $S^0 \times B \to \bar{E} \times \hat{\bar{E}}$.

2.2 An equivalent construction

Next, I'll describe an equivalent construction using Dwyer, Weiss and Williams's description. As before, we at first describe the construction for $M \to *$. We can then generalise canonical constructions to the general case.

Consider the fibrewise one point compactification $TM^\bullet$ of the tangent bundle of M. Its fibre at $x \in M$ is $T_xM^\bullet \cong S^n$. Denote by s_0, s_1 its zero section and ∞ section. Take s_0 as a coherent choice of basepoints. We can replace each fibre of $TM^\bullet$ with the infinite loop space associated to it in a coherent way, i.e. there is a fibre bundle $Q^\bullet(TM)$ over M with $Q^\bullet(T_xM) = \Omega^\infty\Sigma^\infty(T_xM^\bullet)$ as fibre at x, and this construction is functorial.

The Becker-Euler class is the vertical homotopy class represented by s_1. Then the image of it under the following composition, which can be identified as Poincaré duality, is canonically homotopy to the Becker transfer.

$$s_1 = b_n \in \Gamma(Q^\bullet(TM)) \xrightarrow[Thom]{\simeq} \mathrm{Map}(\Sigma^{-w}\,\mathrm{Th}(NM), \mathbf{S^0}) \xrightarrow[SW-duality]{\simeq} \mathrm{Map}_{*/}(S^0, Q^\bullet M) = Q^\bullet(M).$$

For compact manifold with boundary, we have two Becker-Euler classes due to the orientations.

For a general bundle $E \to B$, for each fibre E_b, we have a space of sections of $Q^\bullet(TE_b)$. We need to specify a way to glue these spaces of sections together, so that the parametrised Becker-Euler class is a continuous section $B \to Q^\bullet_B(E)$ of the fibration over B with fibre being the space of torsions.

2.3 Homotopy limit

Following Bousfield and Kan, we think of a diagram of spaces as "a local coefficient system". Homotopy colimit provides a way of gluing. See [2]. To be precise,

Lemma 2.1 ([4])**.** Let $\gamma : I \to \mathbf{CGH}$ be a diagram whose arrows are all homotopy equivalences. Then $\mathrm{hocolim}\,\gamma \to |I|$ is a quasifibration, and $\mathrm{holim}\,\gamma \simeq \mathrm{Map}_{/|I|}(|I|, \mathrm{hocolim}\,\gamma \times^h_{|I|} |I|)$.

A manifold M is identified as a poset $\mathbf{O}(M)$ of its open subsets that are homeomorphic to $\mathbb{R}^n$. Noticing that $\mathrm{hocolim}\, Q^\bullet|_{\mathbf{O}(M)} \simeq Q^\bullet(TM)$, we have $\mathrm{Map}_{/M}(M, Q^\bullet(TM)) \simeq \mathrm{holim}\, Q^\bullet|_{\mathbf{O}(M)}$. Let $E \to |B|$ be a fibre bundle with closed n-manifold fibres. Then parametrised Becker class $b_n(p)$ is by definition a point in $\mathrm{holim}_{(\sigma,\theta)}\, \mathrm{holim}\, Q^\bullet|_{\mathbf{O}(E_\sigma/\theta)}$. The fibrewise Poincaré dual of $b_n(p)$ is canonically homotopy to Becker transfer, i.e. the following diagram commutes up to homotopy, where the right vertical arrow is induced by universal property of homotopy colimit and the lower horizontal arrow is the transfer map

$$\begin{array}{ccc} B & \longrightarrow & Q^\bullet_B(E) \\ \downarrow & & \downarrow \\ B_+ & \longrightarrow & E_+ \end{array}$$

In this sense, this construction is equivalent to Becker-Gottlieb construction. There are at least two advantages of this construction, the first is that we can replace $Q^\bullet$ with any proexcisive functor F, for example the controlled A-theory functor $A^{\%}$, and the second is that b_n constructed in this way admits a combinatorial model. Controlled means additional geometric restrictions, for this case, we can take it as that there is a chain of weak homotopy equivalence between $A^{\%}$ and $\Omega^\infty(-^\bullet \wedge \mathbf{A}^{\%}(*))$. That completes the first part.

2.4 Construction of the parametrised torsion

Now let's move to the second part of my talk which is about the construction of the parametrised torsion.

The unit transformation allows us to relate $Q^\bullet$ to any proexcisive functor in the following way. There is a natural transformation η up to chain of weak homotopy equivalence from $Q^\bullet$ to F. Using this, $b_n(p)$ is carried into a point in $\operatorname{holim}_{(\sigma,\theta)} \operatorname{holim} F|_{\mathbf{O}(E_\sigma/\theta)}$.

Suppose that F can also be applied to E_σ/θ, then the natural transformation given by Poincaré duality $\operatorname{holim} F|_{\mathbf{O}(E_\sigma/\theta)} \to F(E_\sigma/\theta)$ induces a homotopy equivalence

$$\operatorname{holim}_{(\sigma,\theta)} \operatorname{holim} F|_{\mathbf{O}(E_\sigma/\theta)} \to \operatorname{holim}_{(\sigma,\theta)} F(E_\sigma/\theta).$$

Note that $\operatorname{holim}_{(\sigma,\theta)} F(E_\sigma/\theta) \simeq \operatorname{holim}_\sigma F(E_\sigma) \simeq \Gamma(\operatorname{hocolim}_\sigma F(E_\sigma) \times^h_{|\operatorname{simp} B|} |\operatorname{simp} B|)$, denoted as $F_B(E)$.

On the other hand, observing that $|\operatorname{simp} B| \simeq \operatorname{hocolim}_\sigma |\operatorname{simp} B/\sigma|$, we have for every functor F, a natural transformation $|\operatorname{simp} B/-| \to F(E\_)$ determines up to homotopy a section of $F_B(E) \to |\operatorname{simp} B|$. Such natural transformation is called a characteristic.

The proexcisive functor $A^{\%}$ with its characteristic χ connects $Q^\bullet$ and A in the following ways,

(a) the image of $b_n(p)$ under the unit transformation η is connected to $\chi(p)$ by a canonical path if there's an embedding over B of the n-disk bundle over E into the vertical tangent bundle over E;

(b) there is a natural transformation $A^{\%} \to A$ and characteristic χ_h for A s.t. the diagram commutes up to homotopy.

$$\begin{array}{ccc} B & \xrightarrow{\chi(p)} & A^{\%}_B(E) \\ & \searrow^{\chi_h(p)} & \downarrow{\scriptstyle\alpha} \\ & & A_B(E) \end{array}$$

Note that $\chi_h(p)$ is a homotopy invariant.

Let R be a commutative ring with unit, $E \to |B|$ be a fibre bundle. Let $\mathbf{C}(R)$ be the category of bounded chain complexes of finitely generated projective left R-modules. Then, $K(\mathbf{C}(R)) \simeq K(R)$. Let V be a fibre bundle over E whose fibres admit a continuous finitely generated projective left R-module structure. Then the functor $\operatorname{simp} B \to w\mathbf{C}(R)$, carrying simplex σ of B to the singular chain complex $S_*(E_\sigma; V)$ of E_σ with coefficients in V, determines up to homotopy a map $c_V : |B| \simeq |\operatorname{simp} B| \to K(\mathbf{C}(R)) \simeq K(R)$. On the other hand, there is an almost exact functor $\lambda_V : \mathbf{R}_{fd}(E) \to \mathbf{C}(R)$, carrying $s : E \rightleftarrows X : r$ to the singular chain complex of $(X, s(E))$ with coefficients in r^*V.

The construction can be summarised as the following two diagrams commutative up to homotopy,

$$\begin{array}{c} \chi(p) \simeq \mathcal{P}e_n \\ \downarrow{\scriptstyle\simeq} \\ \eta_* \operatorname{tr}_p \simeq \mathcal{P}\eta_* b_n \end{array} \qquad \begin{array}{ccccccccc} B & \xrightarrow{\operatorname{tr}_p} & Q^\bullet_B(E) & \xrightarrow{\eta_*} & A^{\%}_B(E) & \xrightarrow{\alpha} & A_B(E) & \longrightarrow & A(E) \\ & \searrow_{c_V = S_*(-;V)} & & \searrow & & & \downarrow & \swarrow_{linearisation} & \\ & & & & & & K(R) & & \end{array}$$

$\operatorname{hofib}(Q^\bullet_B(E) \to K(R))$ is the space of smooth Whitehead torsions. If c_V is contractible, then by universal property of the kernel, there is a fibrewise lifting

$$B \to \operatorname{hofib}(Q^\bullet(E) \to K(R))_B(E),$$

unique up to homotopy, called the smooth parametrised torsion, which can also be identified as a section of the fibration of the space of torsions. Replacing $Q^\bullet$ with A, we obtain the homotopy parametrised torsion.

2.5 Justification of the name

Put $E \to B$ to be $M \to *$, where M is a compact smooth manifold. Then for a homotopy equivalence $M' \to M$, we have a diagram commutative up to homotopy

$$\begin{array}{ccc} \chi(M) \in A^{\%}(M) & \xrightarrow{\alpha} & A(M) \ni \chi_h(M) \\ \uparrow & & \simeq \uparrow \\ \chi(M') \in A^{\%}(M') & \xrightarrow{\alpha} & A(M') \ni \chi_h(M') \end{array}$$

[3] showed that $\alpha\chi(M) \simeq \chi_h(M), \alpha\chi(M') \simeq \chi_h(M'), \chi_h(M') \simeq \chi_h(M)$, which can be identified as a point in $\mathrm{holim}(A^{\%}(M) \to A(M) \leftarrow A^{\%}(M')$, whose 0-th homotopy group is the Whitehead group $\mathrm{Wh}(\pi_1 M)$. Recall that $\mathrm{Wh}(G)$ is by definition $K_1(\mathbb{Z}G)/\{\pm G\}$. In general, a parametrised smooth Whitehead torsion detects whether a fibrewise homotopy equivalence between two fibre bundles is fibrewise homotopy equivalent to a bundle diffeomorphism. Ok, I've explained how the parametrised Whitehead torsion is constructed.

3 A-theory simplicial presheaf

Now I want to describe an A-theory simplicial presheaf. Let B be the open subsets of X that can be written as union of countable closed subsets of X. Recall [5] that there is a functor $\mathrm{sing}_X : \mathbf{CGH}_{/X} \to \mathrm{Fun}(B^{op}, \mathbf{sSet})$, carrying $Y \to X$ to $U \mapsto \mathbf{CGH}_{/X}(U \times \Delta^{-}, Y)$, called the presheaf of sections, which admits a left adjoint $| - |_X$. For nice X, $|N_T|_X \simeq U$. Then, $\mathrm{sing}_X(Y)$ is an $(\infty, 1)$-sheaf. For a projectively cofibrant $(\infty, 1)$-presheaf F, $F \to \mathrm{sing}_X |F|_X$ is the sheafification of F, which is a weak equivalence if F is already an $(\infty, 1)$-sheaf.

There is a map $F \circ \mathrm{sing}_B(E)(B) \to \mathrm{sing}_B(F_B(E))(B)$. In fact, $E \simeq \mathrm{hocolim}_\sigma E_\sigma$, thus, $\mathrm{sing}_B(E)(B) \simeq \mathrm{holim}_\sigma E_\sigma$. Composing with F, we have $F(\mathrm{sing}_B(E)(B)) \to \mathrm{holim}_\sigma F(E_\sigma) = \mathrm{sing}_B(F_B(E))(B)$.

The idea is to identify the parametrised torsion as a global section of a sheaf of Whitehead spaces.

That concludes my talk for today. Thank you for listening.

References

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