Serre spectral sequence

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What I would like to describe today is Leray-Serre spectral sequence. Spectral sequence was invented to handle the complicated relationship among homology or cohomology or homotopy groups of spaces, when long exact sequence is not powerful enough. The exactness for pairs is generalised to convergence for spectral sequence of a filtered complex and excision is generalised to a covering.

First, let us recall some definitions.

Definition 1. An exact couple is by definition a long exact sequence of the type $\to A \xrightarrow{i} A \xrightarrow{j} E \xrightarrow{k} A \to$.

Lemma 2. Every exact couple gives rises to a derived exact couple $\to A' \xrightarrow{i'} A' \xrightarrow{j'} E' \xrightarrow{k'} A' \to$, where $A' = i(A)$, i' is the restriction of i, $j'(ia) = j(a)$, $E' = \ker(jk)/\operatorname{im}(jk)$ is a subquotient of E, k' is the induced map of the restriction of k.

Definition 3. Let R be a commutative ring. A homologically graded spectral sequence is a sequence of $\mathbb{Z}$-bigraded R-modules $\{E^r\}$ with differential $d^r : E^r \to E^r$ of degree $(-r, r-1)$, s.t. $E^{r+1} = H_*(E^r, d^r)$. r is called the page number. Here a $\mathbb{Z}$-bigraded R-module is a family of R-module indexed over tuples in $\mathbb{Z}^2$.

Dually, we can define a cohomologically graded spectral sequence. We can see from the definition that each page is a family of long exact sequences sharing the same differential d^r.

Exact couples provide a general source for spectral sequences. Filtered complexes give rise to graded exact couples and thus spectral sequences. Instead of giving the general definition for a filtered complex, we examine the following example.

Example 4. *Let X be a topological space and $X^0 \subset X^1 \subset \cdots$ be a sequence of subspaces of X. Then we have an exact couple*

$$\oplus_{n,p} H_n(X^p; G) \xrightarrow{inclusion} \oplus_{n,p} H_n(X^p; G) \xrightarrow{[-]} \oplus_{n,p} H_n(X^p, X^{p-1}; G) \xrightarrow{\partial} \oplus_{n,p} H_n(X^p; G).$$

Denote $A^1_{n,p} = H_n(X^p; G)$ and $E^1_{n,p} = H_n(X^p, X^{p-1}; G)$. The long exact sequence above can be decomposed into

$$A^1_{n,p-1} \xrightarrow{i} A^1_{n,p} \xrightarrow{j=[-]} E^1_{n,p} \xrightarrow{k=\partial} A^1_{n-1,p-1}. \tag{*}$$

Denote $d_1 = jk$, $E^2_{n,p} = \ker d / \operatorname{im} d$ and $A^2_{n,p} = i(A^1_{n,p-1})$. Then applying lemma 2, we have a long exact sequence

$$A^2_{n,p} \xrightarrow{i'} A^2_{n,p+1} \xrightarrow{j'} E^2_{n,p} \xrightarrow{k'} A^2_{n-1,p-1}.$$

Note that $\deg i' = \deg i$, $\deg j' = -\deg i$, $\deg k' = \deg k$. *Reiterating this process, we have*

$$A^r_{n,p+r-2} \to A^r_{n,p+r-1} \to E^r_{n,p} \to A^r_{n-1,p-1}, \tag{**}$$

with $\deg d^r = (-1, -r)$. *Note that* $A^r_{n,p}$ *is a subobject of* $A^1_{n,p}$ *and* $E^r_{n,p}$ *is a subquotient of* $E^1_{n,p}$. $\{E^r_{n,p}, d^r\}$ *is a spectral sequence. For simplicity of Serre spectral sequence, we reindex* $E^1_{n,p}$ *be letting* $E'^1_{p+q,p} = E^1_{n,p}$, *then* $\deg d^r = (-r, r-1)$.

Lemma 5. Suppose

$$\forall n \, \#\{p | E^1_{n,p} \neq 0\} < \infty, \tag{C1}$$

$$\forall n \exists N \forall p < N_n, A^1_{n,p} = 0, \tag{C2}$$

then,

(a) $\forall n \exists N \forall p > N, A^1_{n,p} = A^1_{n,p+1}, A^1_{n,-p} = A^1_{n,-p-1}$,

(b) $A^r_{n,p} = i^{(r-1)} A^1_{n,p+1-r}$ is 0 for $r \gg 1$,

(c) $\forall n \forall p \exists N \forall r > N, E^r_{n,p} = E^{r+1}_{n,p} = E^\infty_{n,p} \cong i^{(\infty)} A^1_{n,p} / i^{(\infty)} A^1_{n,p-1}$,

(d) $A_{n,\infty} \cong \oplus_p E^\infty_{n,p}$, in which case we say that $\{E^2_{n,p}\}$ converges to $A_{n,\infty}$.

Proof. (a) is a direct consequence of (*). (b) follows directly from (C2). Observing that $E^{r+1}_{n,p}$ is the homology of $E^r_{n+1,p+r} \xrightarrow{d^r} E^r_{n,p} \xrightarrow{d^r} E^r_{n-1,p-r}$, and that the first and the third term are 0 for $r \gg 1$, we have $E^r_{n,p}$ stablises as $r \to \infty$. On the other hand, for $r \gg 1$, (**) turns into

$$0 = E^r_{n+1,p+r-1} \to A^r_{n,p+r-2} \to A^r_{n,p+r-1} \to E^r_{n,p} \to A^r_{n-1,p-1} = 0.$$

Thus, $E^r_{n,p} \cong A^r_{n,p+r-1} / A^r_{n,p+r-2} = i^{(r-1)} A^1_{n,p} / i^{(r-1)} A^1_{n,p-1}$. (d) follows from (a) and (c).

Definition 6. Fix a convenient topological category. Recall that a fibration is a continuous map between topological spaces satisfying the homotopy lifting property for $K \to K \times I$ for every topological space K.

Now we can state the main theorem of my talk and of Serre's celebrated paper[4].

Theorem 7 (Serre spectral sequence for homology)**.** *Let* B *be a* CW*-complex and* B^p *be its* p*-skeleton. Let* $F \to X \to B$ *be a Hurewicz fibration, and* $\{E^1_{p,q} = H_{p+q}(X^p, X^{p-1}; G)\}$ *be the spectral sequence induced by filtration* $X^0 \subset X^1 \subset \cdots \subset X^p = \pi^{-1}(B^p) \subset$. *Then,*

(a) $\{E^1_{p,q} = H_{p+q}(X^p, X^{p-1}; G)\}$ *converges to* $H_*(X; G)$,

(b) $(E^1_{p,q}, d^1)$ *is isomorphic to the cellular chain of* B *with coefficients in local system* $H_q(F; G)$,

(c) $E^2_{p,q} \cong H_p(B, \mathscr{H}_q(F; G))$.

And this spectral sequence is natural with respect to maps between fibrations.

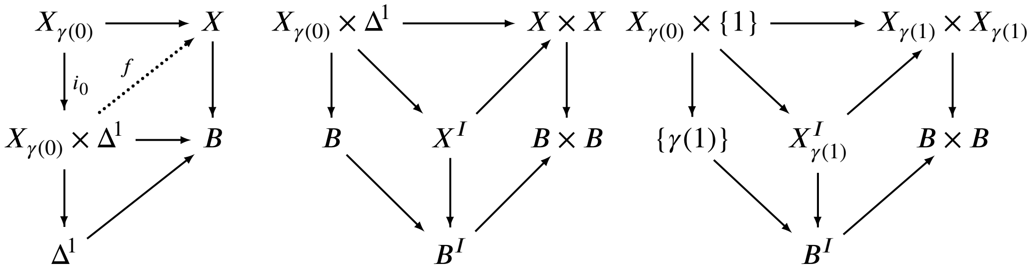
Proof. The proof can be divided into two parts. In the first part, we show that (C1) and (C2) are satisfied and conclude that the spectral sequence stated converges by applying lemma 5. In the second, we define an action on $H_q(F; G)$ and then show that $E^1_{p,q}$ is isomorphic to the group of p-chains of B with coeffients in local system $H_q(F; G)$.

Part 1: It suffices to show that $H_n(X^p; G) \cong H_n(X; G)$ for $n < p$. Since $B^p \to B$ is embedding, $X^p \cong B^p \times_B X$ and thus, $X^p \to B^p$ is a fibration. By naturality of long exact sequence of homotopy groups, we have

$$\begin{array}{ccccccccc} \longrightarrow & \pi_i(F) & \longrightarrow & \pi_i(X^p) & \longrightarrow & \pi_i(B^p) & \longrightarrow & \pi_i(F) & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & \pi_i(F) & \longrightarrow & \pi_i(X) & \longrightarrow & \pi_i(B) & \longrightarrow & \pi_i(F) & \longrightarrow \end{array}$$

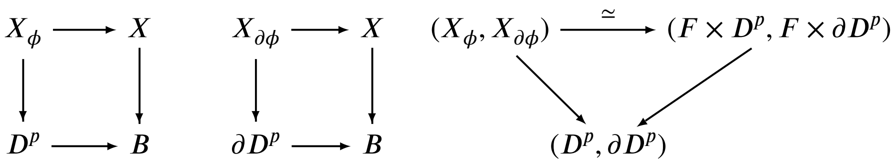
Observing that (B, B^p) is p-connected, by the Four Lemma, we have (X, X^p) is also p-connected. Thus, it has a CW approxiamtion (Y, Y_0) with no cell of dimension lower than $p + 1$. Using cellular homology, we conclude that $H_n(Y; G) \cong H_n(Y_0; G)$ for $n < p$.

Part 2: Denote by X_b the fibre at b, and ΠB the fundamental groupoid of B. First, we construct a functor $\Pi B \to \mathbf{Ab}$ carrying b to $H_q(X_b)$, and a path to the translation of fibres it induces. Let $\gamma : \Delta^1 \to B$ be a path in B. Then the following lifting problem has a unique solution up to homotopy over B. The restriction of f on $X_{\gamma(0)} \times \{1\}$ defines a map $X_{\gamma(0)} \to X_{\gamma(1)}$, and any two such maps are homotopic to each other, thus, induce the same map on homology $H_q(X_{\gamma(0)}; G) \to H_q(X_{\gamma(1)}; G)$. Similarly, for two paths that are homotopic to each other, the induced map on homology are the same. Now choose a basepoint $b \in B$ and a homotopy equivalence $X_b \simeq F$. Restricting this functor to the groupoid at b and modulo the homotopy equivalence class of loops, we have an action of $\pi_1(B, b)$ on $H_q(F; G)$.



Recall that given a local system $\mathscr{H} : \Pi B \to \mathbf{Ab}$, every p-cell $D^p \xrightarrow{f} B$ induces $f^*\mathscr{H} : \Pi D^p \to \Pi B \to \mathbf{Ab}$. Since $\mathscr{H}$ is locally constant, $f^*\mathscr{H}$ admits a direct limit, denoted as $\mathscr{H}_f$. Let $\{(e^p_\alpha, f_\alpha : D^p \to B)\}$ be p-cells of B. Then $\{\bigoplus_{e^p_\alpha} \mathscr{H}_{f_\alpha}\}_p$ is the cellular chain of B with coefficients in $\mathscr{H}$. The boundary operator is defined in the same way as normal cellular homology.

Next, we show that $H_{p+q}(X^p, X^{p-1}; G) \cong \bigoplus_{e^p_\alpha} \mathscr{H}_{f_\alpha}$. Let $(\phi, \partial\phi) : (D^p, \partial D^p) \to (B^p, B^{p-1})$ be a p-cell of B. Then we have pullback bundles



Since D^p is contractible and paracompact, fibration over it is fibre homotopy equivalent to the trivial fibration, i.e. there is a homotopy of pairs over $(D^p, \partial D^p)$ as above. Applying Künneth formula, we have

$$H_{p+q}(X_\phi, X_{\partial\phi}) \cong H_{p+q}(X_b \times D^p, X_b \times \partial D^p) \cong H_q(X_b; G).$$

Recall that a triad (X, A, B) is excisive if $X = \operatorname{Int} A \cup \operatorname{Int} B$. Suppose (B, B_1, B_2) is excisive, then for any $\pi : X \to B$, $(\pi^{-1}(B), \pi^{-1}(B_1), \pi^{-1}(B_2))$ is also excisive, since $\pi^{-1}(\operatorname{Int} U) \subset \operatorname{Int} \pi^{-1}(U)$.

Now, since B is a CW complex, there is an excisive triads (B, B_1, B_2) for every p-cell, s.t. $B_1 \simeq B^{p-1}$, $B_2 \simeq \coprod_{e_\alpha^p} \operatorname{Int} D^p$. From excision axiom and additivity, we have $H_*(X^p, \pi^{-1}(B_1))$.

$$H_{p+q}(X^p, X^{p-1}; G) \cong H_*(X^p, \pi^{-1}(B_1)) \cong \oplus_{e_\alpha^p} H_*(X_\phi, X_{\partial\phi}) \cong \oplus_{e_\alpha^p} \mathscr{H}_{f_\alpha}.$$

And they have the same boundary operator.

Remark. Use CW approximation, we can show that the statement also holds if B is not a CW complex. We can also deduce from the proof that the CW structure of B doesn't matter.

Remark. The Serre spectral sequence for cohomology also captures the multiplicative structure of the cohomology ring, in the sense that it converges as algebra to the coholomogy of the total space with coefficients in a commutative ring. Recall that a cohomologically graded spectral sequence of algebras over a commutative ring R is a cohomologically graded spectral sequence with a compatible family of multiplicative structures making the following diagram commutes.

$$\begin{array}{ccccc} E_{r+1} \otimes_R E_{r+1} & \xrightarrow{\cong} & H(E_r) \otimes H(E_r) & \longrightarrow & H(E_r \otimes_R E_r) \\ \Big\downarrow{\scriptstyle \cdot_{r+1}} & & & & \Big\downarrow{\scriptstyle H(\cdot_r)} \\ E_{r+1} & & \xrightarrow{\qquad\qquad \cong \qquad\qquad} & & H(E_r) \end{array}$$

The spectral sequences used by Leray, Koszul, Cartan were actually spectral sequence of algebras. However, they didn't use singular cohomology. It is Serre who generalised the spectral sequence and unlock the computational power of it. Serre spectral sequence for singular cohomology also captures the multiplicative structure of the cohomology ring. A lot of classical results can be obtained from Leray-Serre spectral sequence, such as, Künneth formula, Hurewicz theorem and Gysin sequence.

Example 8. *Consider fibration $\Omega S^n \to PS^n \to S^n$, where $n > 1$. Then on the second page only the 0-th and n-th column is nonzero. Recall that $\deg d^r = (-r, r-1)$. Thus, $d^r = 0$ for $r \neq n$. Since PS^n is contractible, the infinity page is 0 except at $(0,0)$. Thus, d_n is isomorphism except at $(0,0)$. We conclude that $H_m(\Omega S^n)$ is zero if $n-1 \nmid m$, and that $H_{k(n-1)}(\Omega S^n) \cong H_{n-1}(\Omega S^n)$.*

Example 9. *Consider fibration $K(\mathbb{Z},1) \to PK(\mathbb{Z},2) \to K(\mathbb{Z},2)$. Note that $K(\mathbb{Z},1) \simeq S^1$ and $K(\mathbb{Z},1) \simeq \mathbb{C}P^\infty$. Since $H_q(S^1) = 0$ for $q \neq 0,1$, the second page of the spectral sequence of this fibration is zero except in the 0-th and 1-th rows. Observing that $\deg d^r = (-r, r-1)$, we have $d^r = 0$ for $r > 2$ and $E^\infty_{p,q} \cong E^3_{p,q}$. On the other hand, $E^r_{p,q} \Rightarrow H_*(PK(\mathbb{Z},2))$, thus, $E^\infty_{p,q} = 0$ for $(p,q) \neq (0,0)$. We conclude that d^2 is isomorphism except at $(0,0)$ and $H_q(\mathbb{C}P^\infty) \cong H_{q-2}(\mathbb{C}P^\infty)$ for $q > 1$. Thus, $H_q(\mathbb{C}P^\infty) = 0$ for q odd, and $\mathbb{Z}$ for q even.*

Another important theoretical application of Serre spectral sequence is the Hurewicz theorem for Serre class. People didn't even know whether the spheres have finitely generated homotopy groups. Serre used his spectral sequence to successfully compute the torsion part of a family of homotopy groups of spheres.

Definition 10. Let C be one of the following subsets of abelian groups:

(a) finitely generated abelian groups;

(b) torsion groups s.t. every element has order only divisible by primes in a given set of primes;

(c) finite torsion groups.

Theorem 11 (Hurewicz)**.** *Let X be a simply connected topological space. Suppose $\pi_(X) \in C$ for $i < n$, then the Hurewicz morphism $\pi_n(X) \to H_n(X)$ is an isomorphism mod C.*

Corollary 12. *Let X be a simply connected topological space. The followings are equivalent:*

(a) $\pi_n(X) \in C$ *for all n;*

(b) $H_n(X) \in C$ *for all n;*

The following two lemmas will be used in the proof.

Lemma 13. Let $F \to X \to B$ be a Hurewicz fibration. Suppose $\pi_1(B)$ acts trivially on $H_*(F)$, then every two of the following statements imply the third,

(a) $H_n(X) \in C$ for all n;

(b) $H_n(F) \in C$ for all n;

(c) $H_n(B) \in C$ for all n.

Proof. (2)(3)⇒(1) is a direct consequence of the main theorem 7.

(1)(2)⇒(3). We show that $H_k(B) \in C$ by induction on k. Note that $H_k(B) = E^2_{k,0}$ for $k > 0$ and that $E^\infty_{k,0} = E^N_{k,0}$ for $N \gg 1$. From the convergence, we have $E^\infty_{p,q} \in C$ for $(p,q) \neq (0,0)$. We claim that $E^r_{k,0} \in C$ iff $E^{r+1}_{k,0} \in C$ for $r > 1$. Observing that $E^{r+1}_{k,0}$ is by definition the homology of

$$0 = E^r_{k+r,1-r} \to E^r_{k,0} \xrightarrow{d^r} E^r_{k-r,r-1},$$

we have $E^{r+1}_{k,0} \cong \ker d^r$. From the induction hypothesis, we have $E^2_{k-r,r-1} = H_{k-r}(B, H_{r-1}(F)) \in C$. Thus, $\operatorname{im} d^r$ as a subquotient, is also in C. Then the following short exact sequence implies the claim above.

$$0 \to \ker d^r \to E^r_{k,0} \to \operatorname{im} d^r \to 0$$

Lemma 14. Suppose $G \in C$, then $H_k(K(G,n)) \in C$ for all $k, n > 0$.

Proof. Bylemma 13 for fibration $K(G, n-1) \to PK(G,n) \to K(G,n)$, it suffices to show for $n = 1$, which can be shown by choosing a nice model for Eilenberg-Maclane space.

Proof of theorem 11. First, we prove a special case. Let X be simply connected. Suppose $\pi_n(X) \in C$ for all n, then $H_n(X) \in C$ for all n.

Since X is simply connected, let $X_n \to X_{n-1} \to \cdots \to X_2$ be its Postnikov tower. We have $X_2 \simeq K(\pi_2(X), 2)$, $\pi_i(X_n) \cong \pi_i(X)$ for $i \leq n$, $\pi_i(X_n) = 0$ for $i > n$ and $K(\pi_n(X) \to X_n \to X_{n-1}$ is a fibration. From lemma 14, we have $H_*(K(\pi_n(X), n) \in C$ for all n. Applying lemma 13, we have $H_*(X_n) \in C$ for all n.

Next, we prove the theorem. Apply theorem 7 to fibration $K(\pi_n(X), n) \to X_n \to X_{n-1}$. Noticing that $H_q(K(\pi_n(X), n)) = 0$ for $0 < q < n$, we have $H_{n+1}(X_{n-1}) \cong E^2_{n+1,0} = E^{n+1}_{n+1,0} \xrightarrow{d^{n+1}} E^{n+1}_{0,n} = E^2_{0,n} = H_n(K(\pi_n(X), n))$. From the definition, we have exact sequence

$$E^{n+1}_{n+1,0} \to E^{n+1}_{0,n} \to E^{n+2}_{0,n} \to 0.$$

And from the convergence, we have

$$H_n(X_n) \cong \oplus_{0 \leq p \leq n} E^\infty_{p,n-p} = E^\infty_{0,n} \oplus E^\infty_{n,0}.$$

Observing that $H_n(X_{n-1}) \cong E^\infty_{n,0}$ and $E^\infty_{0,n} \cong E^{n+2}_{0,n}$, we can connect these two exact sequence into

$$H_{n+1}(X_{n-1}) \to H_n(K(\pi_n(X), n)) \to H_n(X_n) \to H_n(X_{n-1}) \to 0.$$

By the assumption that $\pi_i(X) \in C$ for $i < n$, X_{n-1} satisfies the assumption in the special case, thus we have $H_*(X_{n-1}) \in C$, making $H_n(K(\pi_n(X), n)) \to H_n(X_n)$ an isomorphism mod C. We conclude that $\pi_n(X) \cong \pi_n(X_n) \cong \pi_n(K(\pi_n(X), n)) \cong H_n(K(\pi_n(X), n)) \xrightarrow{mod\ C} H_n(X_n) \cong H_n(X)$ is an isomorphism mod C.

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