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Kunneth Exact sequence

C_*, D_* are chain complexes over a P.I.D R & C_q is a free R-module for each q.

Set $Z_q = \ker \partial_q$, $B_q = \mathrm{Im}\, \partial_{q+1}$

where $\partial_q : C_q \to C_{q-1}$ & $\partial_{q+1} : C_{q+1} \to C_q$

S.E.S

$$0 \to Z_* \xrightarrow{\partial_Z} C_* \xrightarrow{\partial} B_{*-1} \xrightarrow{\partial_B} 0$$

where $\partial_Z = 0$ & $\partial_B = 0$

Assuming the complex D_* is free we can tensor with our S.E.S and obtain another S.E.S

$$0 \to Z_* \otimes D_* \to C_* \otimes D_* \to B_{*-1} \otimes D_* \to 0$$

Passing to homology gives a long exact sequence

$$\to H_{n+1}(B_{*-1} \otimes D_*) \to H_n(Z_* \otimes D_*) \to H_n(C_* \otimes D_*)$$

$$\to H_n(B_{*-1} \otimes D_*) \to H_{n-1}(Z_* \otimes D_*) \to$$

By def we have

$$(B_{*-1} \otimes D_*)_n = (B_* \otimes D_*)_{n-1}$$

We get

$$\to H_n(B_* \otimes D_*) \xrightarrow{\delta_{n+1}} H_n(Z_* \otimes D_*) \to H_n(C_* \otimes D_*)$$

$$\to H_{n-1}(B_* \otimes D_*) \xrightarrow{\delta_n} H_{n-1}(Z_* \otimes D_*) \to$$

From this we take the S.E.S

$$0 \to \operatorname{coker} \delta_{n+1} \to H_n(C_* \otimes D_*) \to \ker \delta_n \to 0$$

We want to analyse the maps δ_n & δ_{n+1}

We begin by looking at the map

$\delta'_n : B_{n-1} \to Z_{n-1}$, we suspect the inclusion map.

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r, hook] \arrow[d, "\partial_Z = 0"] & C_n \arrow[r, "\partial"] \arrow[d, "\partial"] & B_{n-1} \ni b \arrow[r] \arrow[d, "\partial_B = 0"] \arrow[dll, bend right=60, "\delta'_n"'] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r, hook, "i"'] & C_{n-1} \arrow[r, "\partial"'] & B_{n-2} \arrow[r] & 0
\end{tikzcd}$$

$b \in B_{n-1}$, exact $\Rightarrow \exists\, c \in C_n$
$\partial(c) = b \Rightarrow b \in C_{n-1}$.
the map i is the inclusion map
so we have $b \in Z_{n-1}$

So $b \in B_{n-1}$ maps to $b \in Z_{n-1}$ & δ'_n is the inclusion map.

Now consider the tensor product chain complex.

$$(B_* \otimes D_*, \partial_{B\otimes D})$$

$$\partial_{B\otimes D}(b \otimes d) = (-1)^{|b|}\, b \otimes \partial_D d$$

up to sign, the boundary map for the complex $B_* \otimes D_*$ is $\partial_{B\otimes D} = 1 \otimes \partial_D$

Have the complex

$$(B_* \otimes D_*, 1 \otimes \partial_D) \overset{B \text{ free}}{\Longrightarrow} H(B_* \otimes D_*) \cong B_* \otimes H(D_*)$$

We can say the same for Z_*

$$H(Z_* \otimes D_*) \cong Z_* \otimes H(D_*)$$

Consider the S.E.S

$$0 \to B_* \to Z_* \to H_*(C_*) \to 0$$

We can tensor this with $H_*(D_*)$ but there is no reson to expet this to be a free complex, so we may lose injectivity at the first step.

$$0 \to B_* \otimes H_*(D_*) \to Z_* \otimes H_*(D_*) \to H_*(C_*) \otimes H_*(D_*) \to 0$$

Remembering the definition of the TOR functor & applying it to this sequence

B_*, Z_* are free submodules of C_* & so will not contribute any terms to the TOR components

we get an exact sequence.

$$0 \to \left\{ \bigoplus_{p+q=n-1} TOR(H_p(C_*), H_q(D_*)) \right\} \to B_* \otimes H(D_*) \xrightarrow{\delta_{n-1}} Z_* \otimes H(D_*)$$

$$\to H(C_*) \otimes H(D_*) \to 0$$

By definition $\bigoplus_{p+q=n-1} TOR(H_p(C_*), H_q(D_*))$

$$\cong \ker(B_* \otimes H_*(D_*) \to Z_* \otimes H_*(D_*))$$

$$= \ker(\delta_{n-1})$$

$$\operatorname{coker}(\delta_{n+1}) = \left(Z_* \otimes H_*(D_*) \big/ B_* \otimes H(D_*)\right)_n \cong (H_*(C_*) \otimes H(D_*))_n$$

Putting this together we get:

$$0 \to \bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \to H_n(C_* \otimes D_*)$$

$$\to \bigoplus_{p+q=n-1} TOR(H_p(C_*), H_q(D_*)) \to 0$$