Asteroidal Sets

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Coloured cyclic operads are many things. Just like operads can be thought of as rooted trees, cyclic operads are best pictured as unrooted trees. They are the multi-object version of cyclic operads, the multi-input version of dagger categories, and the multi-directional (or no direction, depending on how you view them) version of coloured operads. They can be used to model wiring diagrams, multispans, and surfaces. There are applications to Grothendieck-Teichmüller theory.

Categories	Dagger cats
Operads	Cyclic ops

This talk gives an introduction to (coloured) cyclic operads. Firstly, a definition is presented, and then some notable examples: multispans, framed little discs, and the surface operad (which is actually an infinity operad). Throughout, the term "cyclic operad" will refer to the coloured version, using monochrome to refer to the non-coloured version if need be.

1 Cyclic operads

We compare to dagger categories

Definition 1.1 (Dagger category)**.** A dagger category is a category $\mathcal{C}$ equipped with a functor $\dagger : \mathcal{C}^{op} \to \mathcal{C}$ which is the identity on objects, and such that for all objects $A, B, C \in \mathcal{C}$ and all morphisms $f : A \to B$ and $g : B \to C$

1. $id_A^\dagger = id_A : A \to A$
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$
3. $f^{\dagger\dagger} = f$

Note that here $f^\dagger$ is known as f adjoint, named after the corresponding adjoint in Hilbert spaces. Hilbert spaces were the inspiration for dagger categories, and are therefore the first example given.

Hilbert spaces have also provided the following terminology for dagger categories.

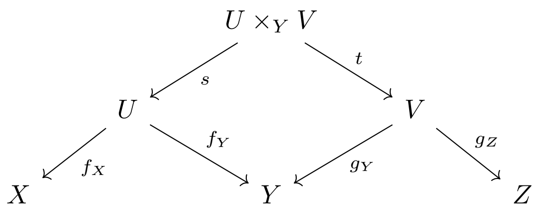
Definition 1.2. A morphism is

- unitary if $f^\dagger = f^{-1}$
- self-adjoint if $f^\dagger = f$
- normal if $f^\dagger f = f f^\dagger$ –this requires $f : X \to X$

Any groupoid is an example of a dagger category, and in particular one in which all the morphisms are unitary. This is because, for any morphism f we can take f^{-1} to be its adjoint.

Definition 1.3 (Span)**.** Let $\mathcal{C}$ be a category with pullbacks, and consider objects in it, X and Y. Then a span between them is an object Z together with morphisms $f : Z \to X$ and $g : Z \to Y$. Alternatively, a span is a diagram $\cdot \leftarrow \cdot \rightarrow \cdot$.

Definition 1.4 (Composition of spans)**.** Let X, Y, Z be objects in a category $\mathcal{C}$ with spans $X \leftarrow U \rightarrow Y$ and $Y \leftarrow V \rightarrow Z$. Then we can find a span from X to Z as follows.



First we take the pullback $U \times_Y V$, then we compose to find a span from X to Z as $(U \times_Y V, f_X \circ s, g_Z \circ t)$

Definition 1.5 (Span category)**.** Let $\mathcal{C}$ be a category with pullbacks. Then $\mathbf{Span}(\mathcal{C})$ is defined as follows:

- $ob(\mathbf{Span}(\mathcal{C})) = ob(\mathcal{C})$
- $mor(A, B) = \{(X, f, g) | f : X \to A, g : X \to B\}$
- Composition of morphisms is given as above
- $id_A = A \leftarrow A \rightarrow A$ where the morphisms are both the identity

There is also the cospan category, formed in a very similar way. We require $\mathcal{C}$ to have pushouts, we have arrows $\cdot \rightarrow \cdot \leftarrow \cdot$, and composition is given by pushouts.

Lemma 1.6. *The span category is a dagger category.*

Proof. Let the adjoint functor $\dagger : \mathbf{Span}(\mathcal{C})^{op} \to \mathbf{Span}(\mathcal{C})$ be defined as the identity on objects, and $(X, f, g) \mapsto (X, g, f)$. Then

1. The identity $(A, id, id)^\dagger = (A, id, id)$
2. Let s and t be the morphisms corresponding to the pullback $F \times_b G$. Then

$$\begin{aligned}
&((G, g_B : G \to B, g_C : G \to C) \circ (F, f_A : F \to A, f_B : F \to B))^\dagger \\
=&(F \times_B G, f_A \circ s, g_C \circ t)^\dagger \\
=&(F \times_B G, g_C \circ t, f_A \circ s) \\
&(F, f_A : F \to A, f_B : F \to B)^\dagger \circ (G, g_B : G \to B, g_C : G \to C)^\dagger \\
=&(F, f_B : F \to B, f_A : F \to A) \circ (G, g_C : G \to C, g_B : G \to B) \\
=&(F \times_B G, g_C \circ t, f_A \circ s)
\end{aligned}$$

3. $(X, f, g)^{\dagger\dagger} = (X, g, f)^{\dagger} = (X, f, g)$

□

The definition of a (coloured) cyclic operad and the definition of a non-cyclic operad differ only in that non-cyclic operads have a distinguished output element, often denoted c_0.

Definition 1.7 (Coloured (symmetric) cyclic operad)**.** A coloured cyclic operad $\mathcal{C}$) is defined with the following properties.

- Let $ob(\mathcal{C})$ be a class of objects, or colours.
- For each profile over $ob(\mathcal{C})$, $\underline{c} = (c_1, \ldots, c_n)$, there is a class $\mathcal{C}(c_1, \ldots c_n)$.
- For any profile $\underline{c}$ and permutation $\sigma \in \Sigma_n$, there is a right action of the symmetric group, a bijection $\mathcal{C}(c_1, \ldots c_n) \to \mathcal{C}(c_{\sigma(1)}, \ldots c_{\sigma(n)})$
- For each colour $c \in ob(\mathcal{C})$ there is an identity element $\eta_c \in \mathcal{C}(c, c)$
- There is a composition operation which identifies two edges of the same colour. It is associative, unital, and equivariant. If $c_i = d_j$, then

$$\mathcal{C}(\underline{c}) \otimes \mathcal{C}(\underline{d}) \to \mathcal{C}(c_1, \ldots \hat{c_i}, \ldots c_m, d_1, \ldots \hat{d_j}, \ldots d_n) =: \mathcal{C}(\underline{cd})$$

$$\theta(\underline{c})_i \circ_j \theta(\underline{d}) \mapsto \theta(c_1, \ldots \hat{c_i}, \ldots c_m, d_1, \ldots \hat{d_j}, \ldots d_n).$$

Then $\mathcal{C}$ is a (coloured symmetric) cyclic operad

Note that this definition of composition seems much closer to the Markl definition of operads than the May definition.

The following are various examples of cyclic operads.

Example 1.8 (Dagger category)**.** Any dagger category $\mathcal{D}$ is a cyclic operad. For each profile $\underline{c}$, $\mathcal{P}(\underline{c})$ is empty unless the length of $\underline{c}$ is 2, and the action of Σ_2 is given by the dagger function.

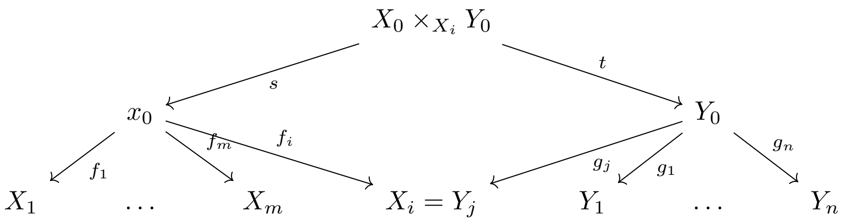
In particular, $Span(\mathcal{C})$ is an example of a cyclic operad.

Example 1.9 (Monochromatic cyclic operad)**.** Any monochromatic cyclic operad, often called just "cyclic operad" in the literature, is a coloured cyclic operad with one colour, hence the term monochromatic.

Example 1.10 (Operad)**.** Given any operad, one can define a cyclic operad by "forgetting" the output. I.e., extending the action of Σ_n to an action of Σ_{n+1}

Definition 1.11 (Multispan)**.** Let $\mathcal{C}$ be a category with pullbacks, and consider objects in it, $X_1, \ldots X_n$. Then a span between them is an object X_0 together with morphisms $f_i : X_0 \to X_i$.

Definition 1.12 (Composition of multispans)**.** Consider spans $(X_1, \ldots X_m; X_0)$ and $Y_1, \ldots Y_n; Y_0)$ such that some $X_i = Y_j$. We denote their morphisms by f_i and g_i, respectively. Then we can find a span $(X_1, \ldots \hat{X_i}, \ldots X_m, Y_1, \ldots \hat{Y_i}, \ldots Y_n; X_0 \times_{X_i} Y_0)$, where $(X_0 \times_{X_i} Y_0, s, t)$ is the pullback of X_0 and Y_0 over $X_i = Y_j$. The internal morphisms will be given by compositions $f_i \circ s$ and $g_i \circ t$.



We say that

$$\begin{aligned}&(X_1, \dots X_m; X_0)_i \circ_j Y_1, \dots Y_n; Y_0)\\ =\ &(X_1, \dots \hat{X}_i, \dots X_m, Y_1, \dots \hat{Y}_i, \dots Y_n; X_0 \times_{X_i} Y_0)\end{aligned}$$

Definition 1.13 (Multispan cyclic operad)**.** Let $\mathcal{C}$ be a category with pullbacks. Then **Multispan**($\mathcal{C}$) is defined as follows:

- The objects, or colours, are $ob(\mathbf{Multispan}(\mathcal{C})) = ob(\mathcal{C})$
- For each profile $\underline{c}$, $\mathcal{C}(\underline{c}) = \{(\underline{c}; c_0) | f_i : c_0 \to c_i, c_0 \in ob(\mathcal{C})\}$
- For any object X, the identity is $(X, X; X)$ where all morphisms are the identity.
- Composition of multispans is as above.
- The action of the symmetric group permutes the order of the objects in the profile.

The multispan cyclic operad is also known as the multicategory of multispans, in the same way that we have the category of spans. In both these cases, we have the extra structure of a dagger category or cyclic operad.

There is also the cospan multicategory, formed in a very similar way. We require $\mathcal{C}$ to have pushouts, we have arrows $\cdot \to \cdot \leftarrow \cdot$, and composition is given by pushouts.

Definition 1.14 (Framed little discs operad (**?**))**.** Let D^n be the open unit disc. Then for all $k \in \mathbb{N}$, let $\mathcal{D}_n(k)$ be the space of embeddings of k disjoint discs into a disc,

$$f : \coprod_k D_n \to D^n$$

where f is a composition of translations, dilations and rotations.

Let f, g_i be such maps. Then composition of maps is defined by compositions of disjoint unions of maps:

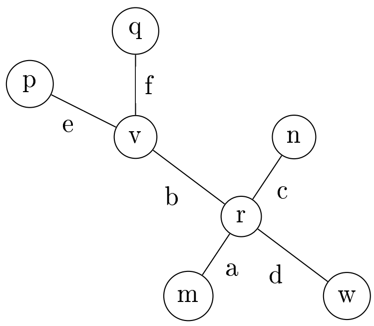
$$\circ(f, g_1, \dots g_k) = \coprod_{n_1 + \dots n_k} D^n \xrightarrow{g_1 \sqcup \dots g_k} \coprod_k D^n \xrightarrow{f} D^n$$

2 The category Ж

To model infinity categories we can use simplicial sets. In the operad case, we can use dendroidal sets. For cyclic operads, the analogous object is an asteroidal set. Therefore, we introduce the category of asterices, or star-like graphs (unrooted trees). The category Ω deals with rooted trees and their associated operads. If we remove the information about the root, we arrive at Ж, the category of unrooted trees.

Definition 2.1 (Unrooted tree)**.** Let G be a simply connected graph. Let O be a subset of the set of outer vertices. Then G, O is an unrooted tree, usually shortened to just G

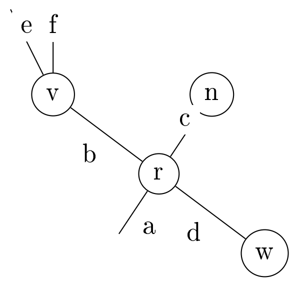
Example 2.2.



The set of inputs is $\{p, q, m\}$. The leaves are $\{e, f, a\}$

As a convention, the input vertices are not drawn, and the word "vertex" refers only to the remaining vertices. Thus, the tree from example 0.2.2 would be drawn as in example 0.2.3.

Example 2.3.

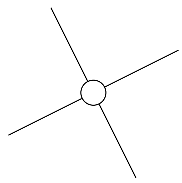


The following trees are important examples.

Definition 2.4 (Corolla - What's the star version)**.** The n-corolla C_n is the tree that has a single vertex v, with n inputs.

Some older sources emphasise the connection to the dendroidal case by writing C_{3+1} rather than C_4

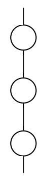
Example 2.5.



This is the 4-corolla

Definition 2.6 (Path - make it cyclic)**.** The n-path P_n is a tree with has n vertices (not including the input and output vertices), and each vertex has exactly one input.

Example 2.7.



This is the 3-path

It is possible to have just a single edge with no vertices. This is the graph η. It is also P_0

Definition 2.8 (η)**.** Consider the graph G defined by $V(G) = \{a, b\}$ and $E(G) = \{\{a, b\}\}$, so

Then $\eta = (G, b, \{a\})$. It is usually drawn

In order to form a category of trees, we must first have morphisms between them. All we want is a graph homomorphism which preserves inputs and output.

Definition 2.9 (Morphisms of trees)**.** Let T and S be rooted trees. Then a morphism $f : T \to S$ consists of

- A function $f_v : V(T) \to V(S)$
- A function $f_e : E(T) \to E(S)$ such that if $e \in E(T)$ is adjacent to $v_1, v_2 \in V(T)$, then $f_e(e)$ is adjacent to $f_v(v_1)$ and $f_v(v_2)$
- If e is an input edge of v then $f_e(e)$ is an input edge of $f_v(v)$

Trees and the maps between them form a category. We call this category Ж, in the same sense that Ω is the category of trees

2.1 Faces, degeneracies, and relations

There are two types of face maps, as well as degeneracy maps. Let T be an unrooted tree. Then

Definition 2.10 (inner face map)**.** Let b be an inner edge of T. Define T/b as the tree created by contracting the edge b. That is, if x and y are the vertices at either side of b, we delete these and the edge b and add a new vertex xy with $nbhd(xy) = (nbhd(x) \cup nbhd(y)) \setminus \{b\}$. Then $\partial_b : T/b \to T$ is the associated inner face map.

Definition 2.11 (outer face map)**.** Let v be an outer vertex of T. Then define T/v to be the tree with v and all its leaves deleted. Then $\partial_b : T/v \to T$ is the associated outer face map

Definition 2.12 (degeneracy map)**.** Let e be any edge of T. Then define T_e to be the graph where the edge e has been subdivided with exactly one vertex. Then $\partial_e : T_e \to T$ is the associated degeneracy map.

Lemma 2.13. *Any morphism of trees can be decomposed into a composition of degeneracy maps followed by isomorphism, then composition of face maps.*

There are some relations too, very similar to those in dendroidal sets. Just like with the tree case, there is a functor which maps the category Ж to cyclic operads. We will state it but prove nothing.

Definition 2.14 (The functor Ж)**.** We define Ж : $\mathit{UnrootedTrees} \to \mathit{cycOp}$ as follows. Given a tree T, we map it to the operad generated by:

- The set of edge labels is the set of colours
- Each vertex v with inputs $\{x_1, x_2, \dots x_n\}$ is an element of $\mathcal{O}(x_1, x_2, \dots x_n)$, i.e. an operation

Additionally, the morphisms are generated by faces and degeneracies.

- Let ∂_e be an inner face map which contracts along edge e between vertices u and v. Then Ж$[\partial_e] : \mathcal{O}(\dots e \dots) \otimes \mathcal{O}(\dots e \dots) \to \mathcal{O}(\dots)$ corresponds to the map $u \circ_e v$ in operads

- Let ∂_v be an outer face map. Then $Ж[\partial_v]$ is the inclusion map $\mathcal{O}(v_1 \dots v_n)\setminus \{v\} \to \mathcal{O}(v_1 \dots v_n)$
- Let σ_e be a degeneracy map. It corresponds to adding an extra operation from e to e

Because of this, we identify the a tree T with the cyclic operad generated from it, $Ж[T]$, and call the category of unrooted trees Ж

3 Asteroidal Sets

Definition 3.1 (Asteroidal sets)**.** $ob(\mathbf{aSet}) = \{\Omega^{op} \to \mathbf{Set}\}$ with natural transformations as morphisms

In other words, an asteroidal set X consists of:

- For each $T \in Ж$, a set $X(T)$, denoted X_T. Note that X_T is called the set of astrices of shape T
- For each morphism $f : S \to T$, a function $X_f : X_T \to X_S$
- Note that $X_{id:T\to T} = id : X_T \to X_T$
- Note that given two morphisms $R \xrightarrow{\beta} S \xrightarrow{\alpha} T$ in Ж, $X_{(\alpha\circ\beta)} = X_\beta \circ X_\alpha$

The definition of kan condition for asteroidal sets is similar to that of dendroidal sets. First, we define the inner horn.

Definition 3.2 (Face)**.** Let $T \in Ж$ be a tree with face map $\alpha : S \to T$. Then the α-face of $Ж[T]$ is the image of the map $Ж[\alpha] : Ж[S] \to Ж[T]$. It is denoted $\partial_\alpha Ж[T]$.

Definition 3.3 (boundary)**.** The asteroidal subset which is the union of all possible faces.

$$\partial Ж[T] = \bigcup_{\alpha\in faces(T)} \partial_\alpha Ж[T]$$

Definition 3.4 (horn)**.** As above, but all except one of the faces

The inner horns are those where the face map which is left out is an inner face map, while the rest are outer horns. More formally,

Definition 3.5 (Inner horn)**.** Let α be an inner face map which removes the edge e. Then

$$\Lambda^e[T] = \bigcup_{\alpha\neq e\in faces(T)} \partial_\alpha Ж[T]$$

In an asteroidal set X, an (inner) horn is a map of dendroidal sets $\Lambda^\alpha[T] \to X$.

Definition 3.6 (inner Kan complex)**.** Let X be an asteroidal set, $f : \Lambda^k[T] \to X$ be an inner horn, and let $j : \Lambda^k[T] \rightarrowtail Ж[T]$ be the inclusion. Then a filler for f is a map $g : Ж[n] \to X$ such that $f = g \circ j$.

$$\begin{array}{ccc} \Lambda^k[T] & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle j} & \nearrow_{g} & \\ \text{Ж}[T] & & \end{array}$$

We say that X is an inner Kan complex if every inner horn has a filler.

If the filler is unique then it is a strict inner Kan complex.