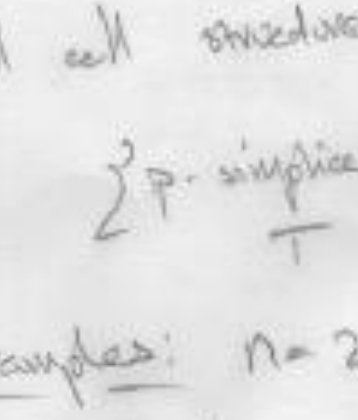
Poincaré Duality Talk

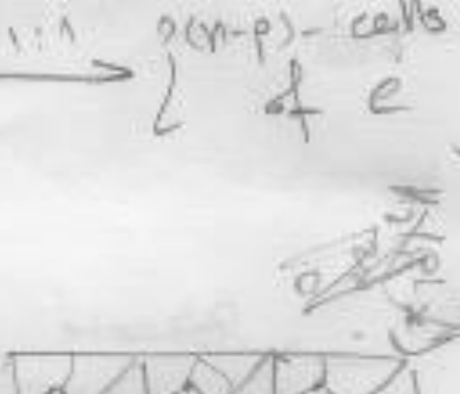
§1

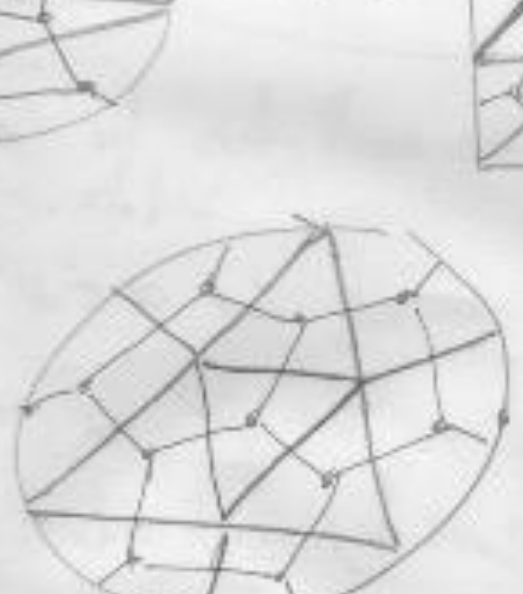
Prop: Let M^n be a compact, connected, n-manifold with a triangulation T. Then T gives a dual cell structure $\mathcal{C}$ on M^n such that

$$\left\{ \begin{matrix} p\text{-simplices of} \\ T \end{matrix} \right\} \xleftrightarrow{1:1} \left\{ \begin{matrix} (n-p)\text{-cells} \\ \text{of } \mathcal{C} \end{matrix} \right\} \quad (*)$$

Examples: $n = 2$.

$S^2 =$ 

$T^2 =$ 

$\mathbb{R}P^2 =$ 

Upshot: We can use the correspondence $(*)$ to build maps $S_p(M^n) \longrightarrow C^{n-p}(M^n)$

$$\rightsquigarrow \begin{array}{ccccc} S_n(M^n) & \to \cdots \to & S_p(M^n) & \to \cdots \to & S_0(M^n) \\ \downarrow f_n & & \downarrow f_p & & \downarrow f_0 \\ C(M^n) & \to \cdots \to & C^{n-p}(M^n) & \to \cdots \to & C^n(M^n) \end{array}$$

<u>claim</u>: If M^n is oreintable then f_* is a chain map and the induced map on homology is an isomorphism

$\rightsquigarrow$ Poincaré Duality:

$$H_p(M^n) \cong H^{n-p}(M^n)$$

§2.

oreintability. (we need this for smooth mflds.)

Let M^n be a n-manifold, Let $x \in M^n$

- A fundamental class of M^n at X is an element $z \in H_n(M^n, M^n - X)$ such that if $x \in X$ then the image of z under the map $H_n(M^n, M^n - X) \to H_n(M^n, M^n - x) \; (\overset{ex.}{\cong} \mathbb{Z})$

$H_n(M^n, M^n - x)$ is a $\mathbb{Z}$-module (i.e. $=1$, or -1)

· An <u>orientation of M</u> is an open cover $M = \bigcup_{i \in I} U_i$ together with a compatible family of fundamental classes

$$\{z_i \in H_n(M^n, M^n - U_i)\}_{i \in I}$$

$$\begin{tikzcd} z_i \\ H_n(M^n, M^n - U_i) \arrow[dr] & \\ & H_n(M^n, M^n - U_i \cap U_j) \\ H_n(M^n, M^n - U_j) \arrow[ur] & \\ z_j \end{tikzcd}$$

<u>Example</u>: $n = 2$, $M^n = \mathbb{P}^1 = U_0 \cup U_1$

where $U_0 = \mathbb{C}$, $U_1 = \mathbb{C}^\times \cup \{\infty\}$.

Now $H_2(\mathbb{P}^1, \mathbb{P}^1 - U_0) \cong \mathbb{Z} \ni z_0$ $\Big\}$ fundamental classes.

$H_2(\mathbb{P}^1, \mathbb{P}^1 - U_1) \cong \mathbb{Z} \ni z_1$

If $x \in U_0$ then $\overset{z_0}{\hat{\mathbb{Z}}} \cong H_2(\mathbb{P}^1, \mathbb{P}^1 - U_0) \to H_2(\mathbb{P}^1, \mathbb{P}^1 - x) \cong \mathbb{Z}$.

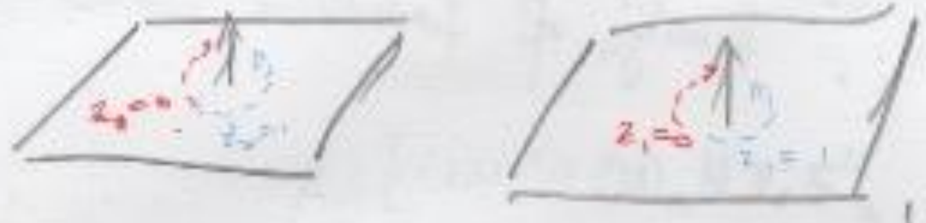
$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\cong"] \arrow[rr, bend right, "\mathrm{id}"'] & H_2(\mathbb{P}^1, \mathbb{P}^1 - U_0) \arrow[r] & H_2(\mathbb{P}^1, \mathbb{P}^1 - x) \cong \mathbb{Z} \end{tikzcd}$$

If $x \in U_1$ then $\overset{z_1}{\hat{\mathbb{Z}}} \cong H_2(\mathbb{P}^1, \mathbb{P}^1 - U_1) \to H_2(\mathbb{P}^1, \mathbb{P}^1 - x) \cong \mathbb{Z}$.

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\cong"] \arrow[rr, bend right, "\mathrm{id}"'] & H_2(\mathbb{P}^1, \mathbb{P}^1 - U_1) \arrow[r] & H_2(\mathbb{P}^1, \mathbb{P}^1 - x) \cong \mathbb{Z} \end{tikzcd}$$

So $z_0 \in \{-1, 1\}$ and $z_1 \in \{-1, 1\}$.

Picture



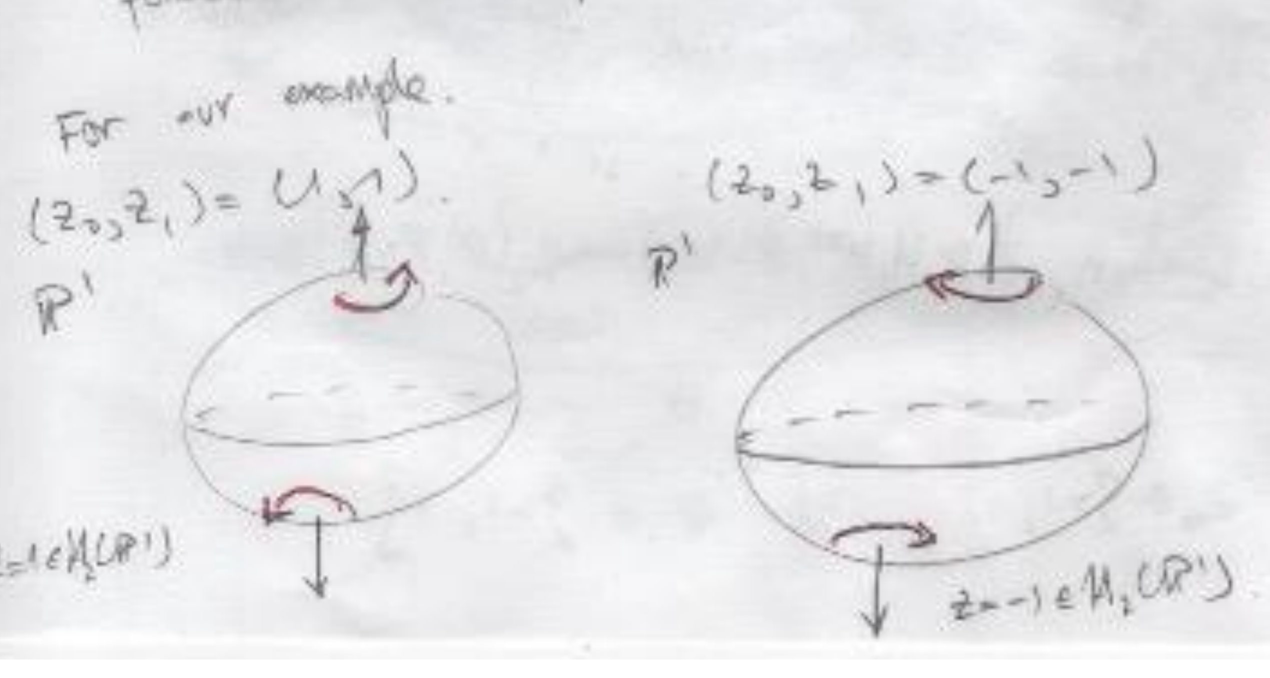
U_0 $\qquad\qquad\qquad\qquad$ U_1.

The compatibility condition boils down to requiring that the two choices of orientation agree after gluing.

<u>Fact 5</u>. A fundamental class of M^n (at M^n) determines an orientation of M^n

. Conversely if $K \subseteq M^n$ is compact then an orientation of M^n at K determines a fundamental class of M^n at K.

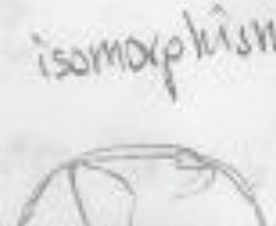
For our example.



Thm: statement of Poincaré Duality.

Thm: Let M^n be a compact oriented n-manifold. If $[M] \in H_n(M^n)$ is a fundamental class of M^n (at M^n) then the map $H^p(M^n) \longrightarrow H_{n-p}(M^n)$ given by $\alpha \longmapsto \alpha \frown [M]$ is a isomorphism.

Proof Philosophy



$\xrightarrow{\cong} \mathbb{R}^2$

For an arbitrary n-manifold we can give a version of Poincaré duality together to get duality for M^n.

Problem: $\mathbb{R}^n$ is not compact.
We need a version of duality for $\mathbb{R}^n$: the cohomology with compact support. Let $\mathcal{K}$ be the set of compact cubes in $\mathbb{R}^n$.

Define $H_c^r(\mathbb{R}^n) := \underset{\mathcal{K}}{\mathrm{colim}}\, H^r(\mathbb{R}^n, \mathbb{R}^n - K)$

structure maps $H^r(\mathbb{R}^n, \mathbb{R}^n - L) \rightarrow H^r(\mathbb{R}^n, \mathbb{R}^n - K)$ come from inclusions $(\mathbb{R}^n, \mathbb{R}^n - L) \subseteq (\mathbb{R}^n, \mathbb{R}^n - K)$.

Goal: Get an isomorphism

$$H^q_c(\mathbb{R}^n) \cong H_{n-q}(\mathbb{R}^n)$$

Universal property of colimits.

$\rightsquigarrow$ It suffices to give isomorphisms

$$D_K : H^q(\mathbb{R}^n, \mathbb{R}^n - K) \longrightarrow H_{n-q}(\mathbb{R}^n)$$

for each compact cube such that if

$K \subseteq L$ then

$$\begin{tikzcd}
H^q(\mathbb{R}^n, \mathbb{R}^n - K) \arrow[dr, "D_K"] & \\
 & H_{n-q}(\mathbb{R}^n) \\
H^q(\mathbb{R}^n, \mathbb{R}^n - L) \arrow[uu] \arrow[ur, "D_L"'] &
\end{tikzcd}$$

How do we get D_K.

Relative cap product

$$\cap : H^q(\mathbb{R}^n, \mathbb{R}^n - K) \otimes_{\mathbb{Z}} H_n(\mathbb{R}^n, \mathbb{R}^n - K) \longrightarrow H_{n-q}(\mathbb{R}^n)$$

Implication

$\mathbb{R}^n$ orientable $\Longrightarrow$ Existence of a fundamental class of $\mathbb{R}^n$ at K $[z]_K \in H_n(\mathbb{R}^n, \mathbb{R}^n - K)$

Define $D_K : H^q(\mathbb{R}^n, \mathbb{R}^n - K) \longrightarrow H_{n-q}(\mathbb{R}^n)$

$$\alpha \longmapsto \alpha \cap [z_K]$$

~~the statement that this an isomorphism is trivial~~

Now $H^q(\mathbb{R}^n, \mathbb{R}^n - K) \cong H^q(\mathbb{R}^n, \mathbb{R}^n - x)$

$$\cong \tilde{H}^{q-1}(\mathbb{R}^n - x)$$

$$\cong \tilde{H}^{q-1}(S^{n-1})$$

$$\cong \tilde{H}^q(S^n)$$

$$= \begin{cases} \mathbb{Z}, & \text{if } q = n \\ 0, & \text{if } q \neq n \end{cases}$$

So D_K is clearly an iso. for $q \neq n$

For $q = n$ this be checked using the definition of $\cap$.