Graduate Topology Seminar W4 : Exercise Session.
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Ex 1. Let $m, n \in \mathbb{N}$. (1) Compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n)$, $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n)$. In particular, show $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = 0$, $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n) = 0$ when m, n are coprime. (2) Compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z})$, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$, $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$.

Hint: Consider the free ($\mathbb{Z}$-) resolution $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0$ of $\mathbb{Z}/m$.

Ex 2: Show that $\mathrm{Ext}^1_{\mathbb{Z}}$, $\mathrm{Tor}_i^{\mathbb{Z}}$ commute with direct sum, in both variables:

For abelian groups A_ν, B_ν, $\nu \in I$ any index set,

$$\mathrm{Ext}^1_{\mathbb{Z}}(\oplus_\nu A_\nu, B) \cong \oplus_\nu \mathrm{Ext}^1_{\mathbb{Z}}(A_\nu, B)$$

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \oplus_\nu B_\nu) \cong \oplus_\nu \mathrm{Ext}^1_{\mathbb{Z}}(A, B_\nu)$$

$$\mathrm{Tor}_i^{\mathbb{Z}}(\oplus A_\nu, B) \cong \oplus_\nu \mathrm{Tor}_i^{\mathbb{Z}}(A_\nu, B),$$

$$\mathrm{Tor}_i^{\mathbb{Z}}(A, \oplus_\nu B_\nu) \cong \oplus_\nu \mathrm{Tor}_i^{\mathbb{Z}}(A, B_\nu),$$

for any abelian groups A, B.

Ex 3: Let X be a finite CW-complex. Show the following:

$$H^n(X;\mathbb{Z}) \cong \mathrm{Fr}(H_n(X;\mathbb{Z})) \oplus \mathrm{Tor}(H_{n-1}(X;\mathbb{Z})).$$

Notations: X being finite, $H^n(X;\mathbb{Z})$ is necessarily a finitely generated abelian group. Therefore we let $\mathrm{Fr}(-)$ denote the free part and $\mathrm{Tor}(-)$ the torsion part.

Proof: By ~~Künneth theorem~~ U.C.T, we have the following short exact sequence:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), \mathbb{Z}) \to H^n(X;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(X;\mathbb{Z}), \mathbb{Z}) \to 0.$$

Everything in sight is a finitely generated abelian group, so by Ex 1 (2) and Ex 2. we have

$$\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X;\mathbb{Z}), \mathbb{Z}) \cong \mathrm{Tor}(H_{n-1}(X,\mathbb{Z}), \mathbb{Z})$$

On the other hand, notice that we have, for any abelian group A.

$$\begin{cases} \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}) \cong A, & A \text{ free} \\ \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}) = 0, & A \text{ torsion.} \end{cases}$$

Therefore $\mathrm{Hom}_{\mathbb{Z}}(H_n(X;\mathbb{Z}), \mathbb{Z}) \cong Fr(H_n(X;\mathbb{Z}))$.

Collecting the above arguments, we have the following exact sequence.

$$0 \to Tor(H_{n-1}(X;\mathbb{Z})) \to H^n(X;\mathbb{Z}) \to Fr(H_n(X;\mathbb{Z})) \to 0$$

Furthermore, it splits since the last term is free. and the desired result follows. ■

Ex 5. Consider the n-torus $T^n := \underbrace{S^1 \times \cdots \times S^1}_{n \text{ copies}}$. Compute $H_*(T^n;\mathbb{Z})$

~~(Including the ring structure)~~

Hint: For $n=2$, we have

$$0 \to \left(H_*(S^1;\mathbb{Z}) \otimes H_*(S^1;\mathbb{Z})\right)_{k} \to H_k(T^2;\mathbb{Z}) \to \sum_{i+j=k-1} Tor(H_i(S^1;\mathbb{Z}), H_j(S^1;\mathbb{Z})) \to 0.$$

The $Tor(-,-)$ term is trivial and we have an isomorphism

$$0 \to \sum_{i+j=k} H_i(S^1;\mathbb{Z}) \otimes H_j(S^1;\mathbb{Z}) \xrightarrow{\cong} H_k(T^2;\mathbb{Z})$$

Now use induction on n.